Journal of Global Optimization (2006) 35: 71–83
DOI 10.1007/s10898-005-1654-1

Existence Results for Systems of Vector Equilibrium Problems*

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(Received 9 June 2004; accepted in revised form 27 July 2005)

Abstract. The purpose of this paper is to study systems of vector equilibrium problems. We establish some existence theorems for systems of vector equilibrium problems by using $(S)_+$-conditions and Kakutani–Fan–Glicksberg fixed point theorem.

Mathematics Subject Classification (2000). 49J40.

Key words: $(S)_+$-conditions, Systems of variational inequalities, Systems of vector equilibrium problems, Vector variational inequality problems

1. Introduction

In the recent years, equilibrium type problems have been well studied. They are related to numerous important subjects of current mathematics, such as, problems of Nash equilibria, optimization problems, variational inequalities, complementarity problems, and fixed point problems, and have been shown to be very useful in mathematical economics, mechanics, numerical analysis and calculus of variations. For details, we refer to [1–32] and the references therein.

Recently, some interesting and important problems related to equilibrium problems and other related problems have been introduced and studied in recent papers. In 1999, Ansari and Yao [1] introduced and studied a system of variational inequalities. In [2], Ansari and Yao introduced and studied systems of generalized variational inequalities. Ansari et al. [3, 4] introduced and studied systems of vector equilibrium problems and gave some applications to vector optimization problems. Very recently, Ansari et al. [5] further introduced and studied a system of vector quasi-equilibrium problems. In the papers [1–5], an important tool is a maximal element theorem due to Deguire et al. [6]. On the other hand, Kassay and Kolumbán [7] introduced a system of variational inequalities and established an existence

*This work was supported by the Kyungnam University Research Fund 2004.

theorem by Ky Fan lemma. In [8], Kassay et al. further introduced and studied Minty and Stampacchia variational inequality systems by Kakutani–Fan–Glicksberg fixed point theorem. In [9], Fang and Huang studied systems of strong implicit vector variational inequalities and proved some existence results by using Kakutani–Fan–Glicksberg fixed point theorem. For other papers related to systems of variational inequalities and complementarity problems, we refer to [10–17] and the references therein. Motivated and inspired by these works, in this paper, we study a system of vector equilibrium problems and prove some existence results by using $(S)_+$-conditions and Kakutani–Fan–Glicksberg fixed point theorem.

The rest of this paper is organized as follows: In Section 2, we give some concepts and notations. In Section 3, we introduce some concepts of $(S)_+$-conditions. Section 4 is devoted to existence results for systems of vector equilibrium problems.

2. Preliminaries and Formulations

In this section, we recall some concepts and give some formulations of problems which will be studied. Let D be a nonempty, closed and convex subset of a real Banach space E and P be a pointed, closed, and convex cone of a real Banach space F with int $P \neq \emptyset$, where int P denotes the interior of P.

DEFINITION 2.1. A mapping $T: D \to 2^F$ (the family of all the nonempty subsets of F) is said to be

(1) upper semi-continuous at $x \in D$ if for any open set V containing $T(x)$, there exists a neighborhood U of x such that $T(U) \subset V$;
(2) upper semi-continuous if T is upper semi-continuous at every $x \in D$;
(3) closed if the graph $Graph\ T = \{(x,u) \in D \times F : u \in T(x)\}$ of T is closed.

Remark 2.1. If the image of T is contained in a compact subset of F, then $T: D \to 2^F$ is upper semi-continuous if and only if T is closed.

DEFINITION 2.2. [18]. A mapping $g: D \to F$ is said to be P-upper semi-continuous if for every $y \in F$, the set $g^{-1}(y - \mathrm{int}P)$ is open in D.

DEFINITION 2.3. [19]. A mapping $g: D \to F$ is said to be

(i) P-convex if

$$tg(x_1) + (1-t)g(x_2) - g(tx_1 + (1-t)x_2) \in \mathrm{int}\ P \cup \{0\}, \quad \forall x_1, x_2 \in D,\ t \in [0,1];$$

(ii) P-quasiconcave if for any $x_1, x_2 \in D, t \in [0, 1]$,

$$g(x_1) \in g(tx_1 + (1-t)x_2) - P \quad \text{or} \quad g(x_2) \in g(tx_1 + (1-t)x_2) - P.$$

DEFINITION 2.4. [19]. A mapping $h: D \times D \to F$ is said to be

(I) P-quasiconvex-like if for any $x, y_1, y_2 \in D, t \in [0, 1]$,

$$h(x, ty_1 + (1-t)y_2) \in h(x, y_1) - P \quad \text{or} \quad h(x, ty_1 + (1-t)y_2) \in h(x, y_2) - P;$$

(II) vector 0-diagonally convex if for any finite set $\{y_1, y_2, \dots, y_n\} \subset D$,

$$\sum_{j=1}^{n} t_j h(x, y_j) \notin -\text{int } P$$

whenever $x = \sum_{j=1}^{n} t_j y_j$ with $t_j \geqslant 0$ and $\sum_{j=1}^{n} t_j = 1$.

EXAMPLE 2.1. Let $E = R, D = R_+, F = R^2, P = R_+^2$, and $h: D \times D \to F$ defined by

$$h(x, y) = \begin{pmatrix} x(y-x)^3 \\ x^3(y-x) \end{pmatrix} \quad \forall x, y \in D.$$

For any finite set $\{y_1, y_2, \dots, y_n\} \subset D$ and $x = \sum_{j=1}^{n} t_j y_j$ with $t_j \geqslant 0$ and $\sum_{j=1}^{n} t_j = 1$, it follows that

$$\sum_{j=1}^{n} t_j h(x, y_j) = \sum_{j=1}^{n} t_j \begin{pmatrix} x(y_j - x)^3 \\ x^3(y_j - x) \end{pmatrix} = \begin{pmatrix} x \sum_{j=1}^{n} t_j (y_j - x)^3 \\ x^3\left(\sum_{j=1}^{n} t_j y_j - x\right) \end{pmatrix}$$
$$= \begin{pmatrix} x \sum_{j=1}^{n} t_j (y_j - x)^3 \\ 0 \end{pmatrix} \notin -\text{int } P.$$

Hence h is vector 0-diagonally convex.

In what follows, unless other specified, we always suppose that I is an index set, for each $i \in I$, K_i is a nonempty, closed and convex subset of a real Banach space X_i, and C_i is a pointed, closed, and convex cone of a real Banach space Y_i with int $C_i \neq \emptyset$. Let $X = \prod_{i \in I} X_i$, $K = \prod_{i \in I} K_i$, $X_{\bar{i}} = \prod_{j \neq i} X_j$, $K_{\bar{i}} = \prod_{j \neq i} K_j$, and for each $i \in I$, $F_i : K_{\bar{i}} \times K_i \times K_i \to Y_i$ be a mapping. The system of vector equilibrium problems is formulated by finding $x=(x_i)_{i\in I}\in K$ such that for all $i\in I$,

$$\text{(SVEP)}\qquad F_i(x_{\bar{i}},x_i,y_i)\notin -\text{int } C_i,\quad \forall y_i\in K_i,$$

where $x_{\bar{i}}=(x_j)_{j\neq i}\in K_{\bar{i}}$.

Special Cases:

(1) If for each $i\in I, Y_i=R, C_i=R_+$ and $F_i=\varphi_i$, where $\varphi_i\colon K_{\bar{i}}\times K_i\times K_i\to R$ is a function, then (SVEP) reduces to the system of equilibrium problems: find $x=(x_i)_{i\in I}\in K$ such that for all $i\in I$,

$$\text{(SEP)}\qquad \varphi_i(x_{\bar{i}},x_i,y_i)\geqslant 0,\quad \forall y_i\in K_i.$$

(2) If for each $i\in I$, $F_i(x_{\bar{i}},x_i,y_i)=\langle T_i(x_{\bar{i}},x_i),y_i-x_i\rangle$, where $T_i\colon K_{\bar{i}}\times K_i\to L(X_i,Y_i)$, and $L(X_i,Y_i)$ denotes the space of all the continuous linear mappings from X_i into Y_i, then (SVEP) reduces the system of vector variational inequality problems: find $x=(x_i)_{i\in I}\in K$ such that for all $i\in I$,

$$\text{(SVVIP)}\qquad \langle T_i(x_{\bar{i}},x_i),y_i-x_i\rangle\notin -\text{int } C_i,\qquad \forall y_i\in K_i.$$

(3) If for each $i\in I, Y_i=R$, and $C_i=R_+$, then (SVVIP) reduces to the system of variational inequality problems: find $x=(x_i)_{i\in I}\in K$ such that for all $i\in I$,

$$\text{(SVIP)}\qquad \langle T_i(x_{\bar{i}},x_i),y_i-x_i\rangle\geqslant 0,\quad \forall y_i\in K_i.$$

(4) If I is a singleton, then (SVEP) reduces to the known vector equilibrium problem (VEP), which also includes as special cases the classical equilibrium problem and variational inequality problem.

Remark 2.2. In terms of maximal element theorems, some existence results for (SVEP), (SEP), (SVVIP), (SVIP) were presented in [1–5], respectively. In [7, 8], some existence results for (SVIP) were proved by Kakutani–Fan–Glicksberg fixed point theorem and Ky Fan lemma when I is a finite set.

3. $(S)_+$-Conditions

In this section, we introduce $(S)_+$-conditions for a family of mappings. First recall some concepts and notations presented in [20–22].

Let Z be a Hausdorff topological vector space, A be a nonempty subset of Z and $C \subset Z$ be a cone with int $C \neq \emptyset$. The superior of A with respect to C is defined by

$$\text{Sup } A = \{z \in \bar{A} : A \cap (z + \text{int } C) = \emptyset\}$$

and the inferior of A with respect to C is defined by

$$\text{Inf } A = \{z \in \bar{A} : A \cap (z - \text{int } C) = \emptyset\},$$

where $\bar{A}$ denotes the closure of A.

As pointed out in [22], the superior Sup A and inferior Inf A with respect to C are extensions of the usual supremum and infimum of A. If A is a nonempty compact subset of Z, then both Sup A and Inf A are nonempty. Let $\{z_\alpha\}_{\alpha \in I}$ be a net in Z. The *limit superior and limit inferior* of $\{z_\alpha\}_{\alpha \in I}$ (with respect to C) are defined by

$$\text{Limsup } z_\alpha = \text{Inf} \bigcup_{\alpha \in I} \text{Sup } S_\alpha \quad \text{Liminf } z_\alpha = \text{Sup} \bigcup_{\alpha \in I} \text{Inf } S_\alpha,$$

where $S_\alpha = \{z_\beta : \beta \succeq \alpha\}$. The *limit superior* and *limit inferior* of $\{z_\alpha\}_{\alpha \in I}$ (with respect to C) are also extensions of the usual limit superior and limit inferior of $\{z_\alpha\}$ (see [22]).

To obtain our main results, we need the following lemma due to Chiang and Yao [22]:

LEMMA 3.1. (Theorem 2.1, [22]). *Let $\{Z_\alpha\}_{\alpha \in I}$ be a net in Z convergent to z, and $S_\alpha = \{z_\beta : \beta \succeq \alpha\}$. Then the following conclusions hold:*

(i) *If there is an α_0 such that for every $\alpha \succeq \alpha_0$ there exists $\beta \succeq \alpha$ with Inf $S_\beta \neq \emptyset$, then $z \in Liminf\, z_\alpha$.*
(ii) *If there is an α_0 such that for every $\alpha \succeq \alpha_0$ there exists $\beta \succeq \alpha$ with Sup $S_\beta \neq \emptyset$, then $z \in Limsup\, z_\alpha$.*

Now we recall some known $(S)_+$-conditions. Let E, F be two Banach spaces, $D \subset F$ be a nonempty set and P be a pointed, closed and convex cone in F. A mapping $T : D \to E^*$ (the dual space of E) is said to be of *class* $(S)_+$ (see [22, 33, 34]) if for any net $\{x_\alpha\} \subset D$,

$$x_\alpha \to x \text{ weakly and } \lim \sup \langle T x_\alpha, x_\alpha - x \rangle \leqslant 0 \Rightarrow x_\alpha \to x \text{ strongly.}$$

In [22], Chiang and Yao extended $(S)_+$-conditions to vectorial mappings. A mapping $T : D \to L(E, F)$ is said to be of *class* $(S)_+$ if for any net $\{x_\alpha\} \subset K$,

$$x_\alpha \to x \text{ weakly and Limsup } \langle T x_\alpha, x_\alpha - x \rangle \subset F \backslash \text{int } P \Rightarrow x_\alpha \to x \text{ strongly.}$$

In [24], Chadli et al. extended $(S)_+$-conditions to bifunctions. Very recently, Fang and Huang [25] further extended $(S)_+$-conditions to vectorial bifunctions. A mapping $\varphi: D \times D \to F$ is said to be of *class* $(S)_+$ if for any net $\{x_\alpha\} \subset D$,

$$x_\alpha \to x \text{ weakly and } \operatorname{Liminf} \varphi(x_\alpha, x) \subset F \backslash (-\text{int } P) \Rightarrow x_\alpha \to x \text{ strongly.}$$

In terms of $(S)_+$-conditions, some existence results for variational inequalities and equilibrium problems were proved in [22–25]. Now we extend $(S)_+$-conditions to a family of mappings.

DEFINITION 3.1. Let $\Phi_i: K \times K_i \to Y_i$ be a mapping for all $i \in I$. We say $\{\Phi_i\}_{i\in I}$ is of *class* $(S)_+$ if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i\in I}\} \subset K$ with weak limit $x = (x_i)_{i\in I} \in K$,

$$\operatorname{Liminf} \Phi_i(x^\alpha, x_i) \subset Y_i \backslash (-\text{int}\, C_i), \quad \forall i \in I \Longrightarrow x^\alpha \to x \text{ strongly.}$$

Remark 3.1.

(1) If for each $i \in I, Y_i = R, C_i = R_+$, then Definition 3.1 reduces to that of a family of functions $\{\varphi_i\}_{i\in I}$, i.e., $\{\varphi_i\}_{i\in I}$ is said to be of *class* $(S)_+$ if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i\in I}\} \subset K$ with weak limit $x = (x_i)_{i\in I} \in K$,

$$\operatorname{Liminf} \varphi_i(x^\alpha, x_i) \geqslant 0, \quad \forall i \in I \Longrightarrow x^\alpha \to x \text{ strongly,}$$

where $\varphi_i: K \times K_i \to R$ is a function.

(2) If for each $i \in I, \Phi_i(x, y_i) = \langle T_i(x), y_i - x_i \rangle$ for all $x = (x_i)_{i\in I} \in K$ and $y_i \in K_i$, then Definition 3.1 reduces to: $\{T\}_{i\in I}$ is said to be of *class* $(S)_+$ if for any net $\{x^\alpha\} = \{(x_i)^\alpha_{i\in I}\} \subset K$ with weak limit $x = (x_i)_{i\in I} \in K$,

$$\operatorname{Limsup} \langle T_i(x^\alpha), x_i^\alpha - x_i \rangle \subset Y_i \backslash \text{int } C_i, \quad \forall i \in I \Longrightarrow x^\alpha \to x \text{ strongly,}$$

where $T_i: K \to L(X_i, Y_i)$.

(3) If I is a singleton, then Definition 3.1 reduces to the definition of $(S)_+$-conditions for vectorial bifunctions in the sense of Fang and Huang [25], which also includes those of other $(S)_+$-conditions in [22–24, 33, 34].

EXAMPLE 3.1. Let K_i be a nonempty, closed, and convex subset of a real reflexive Banach space $X_i, i = 1, 2$, and $\varphi: K_1 \times K_2 \times K_1 \to R$ and $\phi: K_1 \times K_2 \times K_2 \to R$ be two functions, and $\alpha, \beta: R^+ \to R^+$ be two continuous and strictly increasing functions. Assume that

(1) $\varphi(a,b,x)+\varphi(x,b,a)+\alpha(\|a-x\|)\leqslant 0$ and $\phi(a,b,y)+\phi(a,y,b)+\beta(\|b-y\|)\leqslant 0$ hold for all $a,x\in K_1$ and $b,y\in K_2$;
(2) for any fixed $(a,b)\in K_1\times K_2, \varphi(a,\cdot,\cdot)$ and $\phi(\cdot,b,\cdot)$ are completely continuous;
(3) $\varphi(a,b,a)\geqslant 0$ and $\phi(a,b,b)\geqslant 0$ hold for all $(a,b)\in K_1\times K_2$.

Then $\{\varphi,\phi\}$ is of class $(S)_+$.

Proof. Let $\{(a_\lambda,b_\lambda)\}\subset K_1\times K_2$ such that (a_λ,b_λ) converges weakly to (a,b) and

$$\begin{cases}\liminf_\lambda \varphi(a_\lambda,b_\lambda,a)\geqslant 0,\\ \liminf_\lambda \phi(a_\lambda,b_\lambda,b)\geqslant 0.\end{cases}$$

By condition (1),

$$\begin{cases}\varphi(a_\lambda,b_\lambda,a)+\varphi(a,b_\lambda,a_\lambda)+\alpha(\|a_\lambda-a\|)\leqslant 0,\\ \phi(a_\lambda,b_\lambda,b)+\phi(a_\lambda,b,b_\lambda)+\beta(\|b_\lambda-b\|)\leqslant 0.\end{cases}$$

It follows from condition (2) that

$$\begin{cases}\liminf_\lambda \varphi(a_\lambda,b_\lambda,a)+\varphi(a,b,a)+\liminf_\lambda \alpha(\|a_\lambda-a\|)\leqslant 0,\\ \liminf_\lambda \phi(a_\lambda,b_\lambda,b)+\phi(a,b,b)+\liminf_\lambda \beta(\|b_\lambda-b\|)\leqslant 0.\end{cases}$$

By condition (3),

$$\begin{cases}\liminf_\lambda \alpha(\|a_\lambda-a\|)\leqslant 0,\\ \liminf_\lambda \beta(\|b_\lambda-b\|)\leqslant 0.\end{cases}$$

Since α and β are continuous and strictly increasing, (a_λ,b_λ) converges strongly to (a,b). Thus $\{\varphi,\phi\}$ is of class $(S)_+$. □

4. Existence Results

For our main results, we need the following lemma.

LEMMA 4.1. (Lemma 3.1, [25]). *Let D be nonempty, compact, and convex subset of a finite dimensional space E and P be a pointed, closed and convex cone of a real Banach space F with* int $P\neq\emptyset$. *Suppose that $\varphi\colon D\times D\to F$ is a mapping satisfying the following conditions:*

(1) $\varphi(x,x)\notin -int P$ *for all* $x\in D$;
(2) *For every* $y\in D, \varphi(\cdot,y)$ *is P-upper semi-continuous;*
(3) φ *is vector 0-diagonally convex;*
(4) *For every* $y\in D, \varphi(\cdot,y)$ *is P-quasiconcave.*

Then the problem formulated by finding $x\in D$ such that

$$\varphi(x,y)\notin -\operatorname{int} P, \quad \forall y\in D$$

admits a nonempty, bounded, closed and convex solution set.

THEOREM 4.1. *For each* $i\in I$, *let* K_i *be a nonempty, bounded, closed and convex subset of a real reflexive Banach space* X_i *and* C_i *be a pointed, closed, and convex cone of a real Banach space* Y_i *with int* $C_i \neq \emptyset$. *For each* $i\in I$, *let* $F_i\colon K_{\bar{i}}\times K_i\times K_i\to Y_i$ *be a mapping satisfying the following conditions:*

(1) *For any given* $x=(x_i)_{i\in I}$ *and each* $i\in I$, $F_i(x_{\bar{i}},x_i,x_i)\notin -int\, C_i$;
(2) *For any given* $x=(x_i)_{i\in I}$ *and each* $i\in I$, $F_i(x_{\bar{i}},\cdot,x_i)$ *is* C_i*-quasiconcave*;
(3) *For any given* $x=(x_i)_{i\in I}$ *and each* $i\in I$, $F_i(x_{\bar{i}},\cdot,\cdot)$ *is vector* 0*-diagonally convex and* C_i*-upper semicontinuous*;
(4) *For any given* $x=(x_i)_{i\in I}$ *and each* $i\in I$, $F_i(\cdot,\cdot,x_i)$ *is continuous*;
(5) $\{\Phi_i(\cdot,\cdot)\}_{i\in I}$ *is of class* $(S)_+$, *where* $\Phi_i(x,y_i)=F_i(x_{\bar{i}},x_i,y_i)$ *for all* $x=(x_i)_{i\in I}, y_i\in K_i$ *and* $i\in I$;
(6) *For any net* $\{x^\alpha\}=\{(x_i)^\alpha_{i\in I}\}$, *any* $x=(x_i)_{i\in I}\in K$ *and each* $i\in I$, *there exists* α_0 *such that for every* $\alpha\geqslant\alpha_0$ *there exists* $\beta\geqslant\alpha$ *with* $Inf\,\Phi_i(x^\beta,x_i)\neq\emptyset$.

Then (SVEP) is solvable.

Proof. Define $\mathcal{M}$ by

$$\mathcal{M}=\left\{M\subset X: M=\prod_{i\in I}M_i \text{ with } M_i \text{ is a finite-dimension subspace of } X_i \text{ and } K_{M_i}=K_i\cap M_i\neq\emptyset \text{ for all } i\in I\right\}.$$

For given $M\in\mathcal{M}$ and $z=(z_i)_{i\in I}\in K$, consider the following auxiliary problems:

$$(AP)^i_M \qquad \text{find } x_i\in K_{M_i}, \quad \text{such that } F_i(z_{\bar{i}},x_i,y_i)\notin -\text{int } C_i, \quad \forall y_i\in K_{M_i}, \quad i\in I.$$

It is easy to see that for each $i\in I$, $F_i(z_{\bar{i}},\cdot,\cdot)$ satisfies all the assumptions of Lemma 4.1 from conditions (1)–(4). By Lemma 4.1, for each $i\in$

$I, (AP)_M^i$ has a nonempty, bounded, closed and convex solution set. For each $i \in I$, define a multivalued mapping $T_i: K_{M_{\bar{i}}} \to 2^{K_{M_i}}$ by

$$T_i(z_{\bar{i}}) = \{x_i \in K_{M_i} : F_i(z_{\bar{i}}, x_i, y_i) \notin -\text{int}\, C_i, \forall y_i \in K_{M_i}\}, \quad \forall z_{\bar{i}} \in K_{M_{\bar{i}}}.$$

The arguments above imply that for each $i \in I$ and $z_{\bar{i}} \in K_{M_{\bar{i}}}, T_i(z_{\bar{i}})$ is nonempty, bounded, closed and convex. Furthermore, for each $i \in I$, it is easy to verify that T_i has a closed graph from condition (4). So T_i is upper semi-continuous by Remark 2.1. Now define $T: K_M \to 2^{K_M}$ by

$$T(z) = (T_i(z_{\bar{i}}))_{i \in I}, \quad \forall z = (z_i)_{i \in I} \in K_M.$$

From the above arguments, we know that T is upper semi-continuous with nonempty, compact and convex values. By Kakutani–Fan–Glicksberg fixed point theorem (see [35]), there exists $u = (u_i)_{i \in I} \in K_M$ such that $u \in T(u)$, i.e.,

$$u_i \in K_{M_i} \quad \text{and} \quad F_i(u_{\bar{i}}, u_i, y_i) \notin -\text{int}\, C_i, \quad \forall y_i \in K_{M_i}, \ \forall i \in I.$$

Denote by S_M the solution set of the following problem:

$$\text{find } u = (u_i)_{i \in I} \in K \text{ such that } F_i(u_{\bar{i}}, u_i, y_i) \notin -\text{int}\, C_i, \quad \forall y_i \in K_{M_i}, \ \forall i \in I.$$

Obviously S_M is nonempty and bounded. Then $\overline{S}_M^w$ is weak compact since X_i is reflexive for all $i \in I$, where $\overline{S}_M^w$ is the weak closure of S_M in K. Let $M^j = \prod_{i \in I} M_i^j \in \mathcal{M}$, $j = 1, 2, \ldots, n$ and $L = \prod_{i \in I} L_i$ with L_i is the subspace of X_i spanned by $\bigcup_{j=1}^n M_i^j$ for all $i \in I$. It is easy to see that $S_L \subset \bigcap_{j=1}^n S_{M^j}$. This implies that $\{\overline{S}_M^w : M \in \mathcal{M}\}$ has the finite intersection property. It follows that

$$\bigcap_{M \in \mathcal{M}} \overline{S}_M^w \neq \emptyset.$$

Let $u^* = (u_i^*)_{i \in I} \in \bigcap_{M \in \mathcal{M}} \overline{S}_M^w$. We assert that u^* is a solution of (SVEP). For any given $y = (y_i)_{i \in I} \in K$, choose $M \in \mathcal{M}$ such that $u^*, y \in K_M$. Then there exists a net $\{u^\alpha\} = \{(u_i)_{i \in I}^\alpha\} \in S_M$ with weak limit u^* since $u^* \in \overline{S}_M^W$. It follows that for each $i \in I$

$$F_i(u_{\bar{i}}^\alpha, u_i^\alpha, u_i^*) \notin -\text{int}\, C_i, \quad \forall \alpha.$$

Hence

$$\text{Liminf}\, \Phi_i(u^\alpha, u_i^*) \subset Y_i \setminus (-\text{int}\, C_i), \quad \forall i \in I.$$

Since $\{\Phi_i(\cdot,\cdot)\}_{i\in I}$ is of class $(S)_+, u^\alpha$ converges strongly to u^*, i.e., for each $i\in I, u_i^\alpha$ converges strongly to u_i^*. For any given $y=(y_i)_{i\in I}\in K$, from condition (4), we know that for each $i\in I$, $\Phi_i(u^\alpha, y_i)$ converges strongly to $\Phi_i(u^*, y_i)$. By Lemma 3.1 and condition (6),

$$\text{for each } i\in I, \quad \Phi_i(u^*, y_i)\in \operatorname{Liminf}\Phi_i(u^\alpha, y_i)\subset Y_i\setminus(-\operatorname{int}C_i), \quad \forall y_i\in K_i,$$

i.e.,

$$\text{for each } i\in I, \quad F_i(u_{\bar{i}}^*, u_i^*, y_i)\notin -\operatorname{int}C_i, \quad \forall y_i\in K_i.$$

□

By Theorem 4.1, we obtain the following existence results.

COROLLARY 4.1. *For each $i\in I$, let K_i be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X_i and $\varphi_i: K_{\bar{i}}\times K_i\times K_i\to R$ be a function satisfying the following conditions:*

(1) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, \varphi_i(x_{\bar{i}}, x_i, x_i)\geqslant 0$;*
(2) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, \varphi_i(x_{\bar{i}}, \cdot, x_i)$ is quasiconcave;*
(3) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, \varphi_i(x_{\bar{i}}, \cdot, \cdot)$ is 0-diagonally convex and upper semi-continuous;*
(4) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, \varphi_i(\cdot, \cdot, x_i)$ is continuous;*
(5) *$\{\phi_i(\cdot,\cdot)\}_{i\in I}$ is of class $(S)_+$, where $\phi_i(x, y_i)=\varphi_i(x_{\bar{i}}, x_i, y_i)$ for all $x=(x_i)_{i\in I}, y_i\in K_i$, and $i\in I$.*

Then (SEP) is solvable.

COROLLARY 4.2. *For each $i\in I$, let K_i be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X_i and C_i is a pointed, closed, and convex cone of a real Banach space Y_i with $int\, C_i\neq\emptyset$. For each $i\in I$, let $T_i: K_{\bar{i}}\times K_i\to L(X_i, Y_i)$ be a mapping satisfying the following conditions:*

(1) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, y_i\mapsto\langle T_i(x_{\bar{i}}, y_i), x_i-y_i\rangle$ is C_i-quasiconcave;*
(2) *For any given $x=(x_i)_{i\in I}$ and each $i\in I, (z_i, y_i)\mapsto\langle T_i(x_{\bar{i}}, z_i), y_i-z_i\rangle$ is vector 0-diagonally convex and C_i-upper semi-continuous;*
(3) *For each $i\in I, T_i$ is continuous;*
(4) *$\{\bar{T}_i\}_{i\in I}$ is of class $(S)_+$, where $\bar{T}_i(x)=T_i(x_{\bar{i}}, x_i)$ for all $x=(x_i)_{i\in I}$ for all $i\in I$;*
(5) *For any net $\{x^\alpha\}=\{(x_i)_{i\in I}^\alpha\}$, any $x=(x_i)_{i\in I}\in K$ and each $i\in I$, there exists α_0 such that for every $\alpha\geqslant\alpha_0$ there exists, $\beta\geqslant\alpha$ with $Sup\langle\bar{T}_i(x^\beta), x_i-x_i^\beta\rangle\neq\emptyset$.*

Then (SVVIP) is solvable.

COROLLARY 4.3. *For each $i \in I$, let K_i be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X_i and let $T_i: K_{\bar{i}} \times K_i \to X_i^*$ be a mapping satisfying the following conditions:*

(1) For any given $x = (x_i)_{i \in I}$ and each $i \in I$, $y_i \mapsto \langle T_i(x_{\bar{i}}, y_i), x_i - y_i \rangle$ is quasiconcave;
(2) For any given $x = (x_i)_{i \in I}$ and each $i \in I$, $(z_i, y_i) \mapsto \langle T_i(x_{\bar{i}}, z_i), y_i - z_i \rangle$ is 0-diagonally convex and upper semi-continuous;
(3) For each $i \in I$, T_i is continuous;
(4) $\{\bar{T}_i\}_{i \in I}$ is of class $(S)_+$, where $\bar{T}_i(x) = T_i(x_{\bar{i}}, x_i)$ for all $x = (x_i)_{i \in I}$ for all $i \in I$.

Then (SVIP) is solvable.

Remark 4.1. The method used in the proof of Theorem 4.1 is quite different from those in [1–5], where some existence results for (SVEP), (SEP), (SVIP) were also obtained.

5. Acknowledgement

The authors would like to express their thanks to the referees for their helpful comments and suggestions.

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